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THE ARCHIMEDEAN PROPERTY IN AN ORDERED SEMIGROUP

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The detailed version will appear in 'The Journal of the Australian Mathematical Society' under the same title. So here we give only the abstract of the results.

By an ordered semigroup we mean a semigroup with a simple order which is compatible with the semigroup operation. We denote by P a nonnegatively ordered semigroup, that is, an ordered semigroup in which $x \leqq x^2$ for every $x \in P$, and denote by E the set of idempotents of P. Also we denote the D-equivalence of the semigroup E by D_E. In P, we define the archimedean equivalence $\sim$ by

$x \sim y$ if and only if $x \leqq y \leqq x^n$ or $y \leqq x \leqq y^n$ for some positive integer n.

Lemma 1. For an archimedean class A of P, the following conditions are equivalent:

(1) A contains an idempotent,

(2) A has the greatest element,

(3) A has the zero element,

(4) every element of A is an element of finite order,

(5) A contains an element of finite order.

Thus archimedean classes are divided into two types, periodic one and nonperiodic one.

P is called a-regular if the archimedean equivalence in P is a congruence relation.

Theorem 1. In order that P is not a-regular, it is necessary and sufficient that P contains a subsemigroup isomorphic as an ordered semigroup to either K_1 or K_2, where K_1 and K_2 are ordered semigroups consisting of elements $e < f < a < g$ and with the multiplication tables

K_1

	e	f	a	g
e	e	e	e	e
f	f	f	f	f
a	f	g	g	g
g	g	g	g	g

K_2

	e	f	a	g
e	e	f	f	g
f	e	f	g	g
a	e	f	g	g
g	e	f	g	g

.

Theorem 2. P is a-regular if and only if it satisfies the condition:

$a \sim g = g^2$, $e = e^2 < g$ and $e\ D_E\ g$ imply either $ea = g$ or $ae = g$.

In the following, P is an a-regular nonnegatively ordered semigroup. Then the quotient semigroup $\bar{P} = P/\sim$ with the order induced in natural way is an ordered idempotent semigroup. We denote the D-equivalence in $\bar{P}$ by $\bar{D}$ and the $\bar{D}$-class which contains $A \in \bar{P}$ by $\bar{D}(A)$.

Theorem 3. If A is a periodic archimedean class, then every element in $\bar{D}(A)$ is a periodic archimedean class.

Thus $\bar{D}$-class $\bar{D}$ in $\bar{P}$ belongs to one and only one of the following two types:

(1) all archimedean classes in $\bar{D}$ are periodic,

(2) all archimedean classes in $\bar{D}$ are nonperiodic.

If a $\bar{D}$-class $\bar{D}$ is of the type (1) (type (2)), $\bar{D}$ is called a periodic (nonperiodic) $\bar{D}$-class.

Theorem 4. If A is an archimedean class which belongs to a periodic $\bar{D}$-class $\bar{D}$ and if A is not the least element of $\bar{D}$, then every element of A is at most of order 2.

Theorem 5. Every nonperiodic $\bar{D}$-class consists of only one nonperiodic archimedean class.

Theorem 6. Let A and B be archimedean classes such that $A < B$.

(1) If $AB < B$, then AB is a periodic archimedean class and, for every $a \in A$ and $b \in B$, the product ab is equal to the idempotent of AB.

(2) If $BA < B$, then BA is a periodic archimedean class and, for every $a \in A$ and $b \in B$, the product ba is equal to the idempotent of BA.

Addendum ––– Acknowledgements and Problems

1. Kowalski [3] proved the following

Theorem. Let S be a nonperiodic archimedean ordered semigroup. Then there exists an o-homomorphism of S into the additive semigroup of positive numbers such that two distinct elements of S have the same image if and only if they form an anomalous pair. (cf. Fuchs [1] Theorem 6 on p.170).

We generalized the notion of an anomalous pair to what we call a neighboring pair and obtained a generalization of the Kowalski Theorem such as includes a certain kind of periodic archimedean ordered semigroups. It will appear in another paper [4].

2. Problem. Discuss the connection between the notion of an archimedean class in the above sense and that in the algebraic semigroup sense.

3. Problem. Characterize nonnegative cones of inverse ordered semigroups.

4. Problem. Characterize finite regular ordered semigroups

5. Yamada [5] studied a certain kind of regular semigroups, which he called generalized inverse semigroups. Here we remark that an ordered regular semigroup is a generalized inverse semigroup.

References

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